...th the Author's kind regards.

GREEK GEOMETRY,

FROM

THALES TO EUCLID.

PART VI.

Fourth Article.

BY

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GREEK GEOMETRY FROM THALES TO EUCLID.*

VI.

MENAECHMUS—pupil of Eudoxus, associate of Plato, and the discoverer of the conic sections—is rightly considered by Th. H. Martin[1] to be the same as the Manaechmus of Suidas and Eudocia, 'a Platonic philoso-

* It is pleasing to see, as I said in the last number of HERMATHENA, that: 'The number of students of the history of mathematics is ever increasing; and the centres in which the subject is cultivated are becoming more numerous;' and it is particularly gratifying to observe that the subject has at last attracted attention in England. Since the second part of this Paper was published Dr. Heiberg, of Copenhagen, has completed his edition of Archimedes: Archimedis *Opera Omnia cum Commentariis Eutocii.* e codice Florentino recensuit, Latine vertit notisque illustravit J. L. Heiberg, Dr. Phil. vols. ii. et iii.: Lipsiae, 1881. Dr. Heiberg has been since engaged in bringing out, in conjunction with Professor H. Menge, a complete edition of the works of Euclid, of which two volumes have been published: Euclidis *Elementa*, edidit et Latine interpretatus est J. L. Heiberg, Dr. Phil. vol. i., Libros I–IV continens, vol. ii., Libros V–IX continens, Lipsiae, 1883, 1884. As Heiberg's edition of Archimedes was preceded by his *Quaestiones Archimedeae*, Hauniae, 1879; so, in anticipation of his edition of Euclid he has published: *Litterargeschichtliche Studien über Euklid*, Leipzig, 1882, a valuable work, to which I have referred in the last part of this Paper. Dr. Hultsch, of Dresden, informs me that his edition of Autolycus is finished, and that he hopes it will appear at the end of this month (June, 1885). The publication of this work—in itself so important, inasmuch as the Greek text of the propositions only of Autolycus has been hitherto published—will have, moreover, an especial interest with regard to the subject of the pre-Euclidian geometry. The Cambridge Press announce a work by Mr. T. L. Heath (author of the Articles on 'Pappus' and 'Porisms' in the *Encyclopædia Britannica*) on Diophantus; a subject on which M. Paul Tannery also has been occupied for some time.

The following works on the history of Mathematics have been recently published:—

Marie, Maximilien, *Histoire des Sciences Mathématiques et Physiques*, Tomes I–V, Paris 1883, 1884. The first volume alone—*De Thalès à Dio-*

pher of Alopeconnesus; but, according to some, of Proconnesus, who wrote philosophic works and three books

phante—treats of the subject of these Papers. It is, in my judgment, inferior to the *Histoire des Mathématiques* of M. Hoefer, notwithstanding the errors of the latter, to which I called attention in HERMATHENA, vol. iii. p. 161. For the historical part of this volume M. Marie has followed Montucla without making use, or even seeming to suspect the existence, of the copious and valuable materials which have of late years accumulated on this subject. Referring to this, Heiberg (*Philologus* XLIII. *Jahresberichte*, p. 324) says: 'The author has been engaged with his book for forty years: one would have thought rather that the book was written forty years ago.' M. Marie commences his *Preface* by saying: 'The history that I have desired to write is that of the filiation of ideas and of scientific methods;' as if that was not the aim of all recent enlightened inquiries. Hear what Hankel, in *Bullettino Boncompagni*, v. p. 297, *seq.*, says: *La Storia della matematica non deve semplicemente enumerare gli scienzati e i loro lavori, ma essa deve altresi esporre lo sviluppo interno delle idee che vegnano nella scienza* (Quoted by Heiberg in *Philologus*, l. c.).

Gow, James, *A Short History of Greek Mathematics*, Cambridge, 1884. This history, as far at least as geometry is concerned, is not, nor indeed does is pretend to be, a work of independent research. Unlike M. Marie, however, Mr. Gow has to some extent studied the recent works on the subject, and the reader will see that he has made much use of the first and second parts of this Paper. On the other hand, he has left unnoticed many important publications. In particular, the numerous and valuable essays of M. Paul Tannery, which leave scarcely any department of ancient mathematics untouched, and which throw light on all, seem to be altogether unknown to him. Essays and monographs like these of M. Tannery and others are in fact, with the single exception of Cantor's *Vorlesungen über Geschichte der Mathematik*, the only works in which progress in the history of ancient mathematics has of late years been made: Bretschneider's *Geometrie vor Euklides* and Hankel's *Geschichte der Mathematik* are no exceptions; for the former work is a monograph, and the latter, which was interrupted by the death of the author, contains only some fragments of a history of mathematics, and consists in reality of a collection of essays. Should the reader look at Heiberg's Paper in the *Philologus*, XLIII., 1884, pp. 321–346 and pp. 467–522, which has been referred to above, he will see how numerous and how important are the publications on Greek mathematics which have appeared since the opening of a new period of mathematico-historical research with the works of Chasles and Nesselmann more than forty years ago.

A glance at the subjoined list of the Papers of a single writer—M. Paul Tannery—relating to the period from Thales to Euclid, will enable the reader

on Plato's Republic.' From the following anecdote, taken from the writings of the grammarian Serenus and handed

to form an opinion on the extent of the literature treated of by Dr. Heiberg.

Mémoires de la Société des Sciences physiques et naturelles de Bordeaux (2e Série).—Tome I., 1876, Note sur le système astronomique d'Eudoxe. Tome II., 1878, Hippocrate de Chio et la quadrature des lunules; Sur les solutions du problème de Délos par Archytas et par Eudoxe. Tome IV., 1882, De la solution géométrique des problèmes du second degré avant Eudoxe. Tome V., 1883, Seconde note sur le système astronomique d'Eudoxe; Le fragment d'Eudème sur la quadrature des lunules.

Bulletin des Sciences Mathématiques et Astronomiques.—Tome VII., 1883, Notes pour l'histoire des lignes et surfaces courbes dans l'antiquité. Tome IX., 1885, Sur l'Arithmétique Pythagorienne. Le vrai problème de l'histoire des Mathématiques anciennes.

Annales de la facultè des lettres de Bordeaux.—Tome IV., 1882, Sur les fragments d'Eudème de Rhodes relatifs à l'histoire des mathématiques. Tome V., 1883, Un fragment de Speusippe.

Revue philosophique de France et de l'étranger, dirigée par M. Ribot.—Mars, 1880, Thalès et ses emprunts á l'Égypte.

Novembre, 1880, Mars, Août et Décembre, 1881. L'éducation Platonicienne.

[1] Theonis Smyrnaei Platonici *Liber de Astronomiã*, Paris, 1849, p. 59. A. Böckh (*Ueber die vierjährigen Sonnenkreise der Alten*, Berlin, 1863, p. 152), Schiaparelli (*Le Sfere Omocentriche di Eudosso, di Callippo e di Aristotele*, Milano, 1875, p. 7), and Zeller (*Plato and the Older Academy*, p. 554, note (28), E. T.), hold the same opinion as Martin: Bretschneider (*Geom. vor Euklid.*, p. 162), however, though thinking it probable that they were the same, says that the question of their identity cannot be determined with certainty. Martin and Bretschneider, both, identify Menaechmus Alopeconnesius with the one referred to by Theon in the fragment (*k*) given below. Max C. P. Schmidt (*Die fragmente des Mathematikers Menaechmus*, Philologus, Band XLII. p. 77, 1884), on the other hand, holds that they were distinct persons, but says that it is certainly more probable that the Menaechmus referred to by Theon was the discoverer of the conic sections, than that he was the Alopeconnesian, inasmuch as Theon connects him with Callippus, and calls them both μαθηματικοί. Schmidt, however, does not give any reason in support of his opinion that the Alopeconnesian was a distinct person. But when we consider that Alopeconnesus was in the Thracian Chersonese, and not far from Cyzicus, and that Proconnesus, an island in the Propontis, was still nearer to Cyzicus, and that, further, the Menaechmus referred to in the extract (*k*) modified the system of concentric spheres of Eudoxus, the supposition of Th. H. Martin (*l. c.*) that this extract occurred in the work of the Alopeconnesian on Plato's *Republic* in connexion with the distaff of the Fates in the tenth book becomes probable.

down by Stobaeus, he appears to have been the mathematical teacher of Alexander the Great:—Alexander requested the geometer Menaechmus to teach him geometry concisely; but he replied: 'O king, through the country there are private and royal roads, but in geometry there is only one road for all.'[2] We have seen that a similar story is told of Euclid and Ptolemy I. (HERMATHENA, vol. iii. p. 164).

What we know further of Menaechmus is contained in the following eleven fragments:[3]—

(*a*). Eudemus informs us in the passage quoted from Proclus in the first part of this Paper (HERMATHENA, vol. iii. p. 163), that Amyclas of Heraclea, one of Plato's companions, and Menaechmus, a pupil of Eudoxus and also an associate of Plato, and his brother, Dinostratus, made the whole of geometry more perfect.[4]

(*b*). Proclus mentions Menaechmus as having pointed out the two different senses in which the word element, στοιχεῖον, is used.[5]

(*c*). In another passage Proclus, having shown that many so-called conversions are false and are not properly conversions, adds that this fact had not escaped the notice of Menaechmus and Amphinomus and the mathematicians who were their pupils.[6]

(*d*) In a third passage of Proclus, where he discusses

[2] Stobaeus, *Floril.*, ed. A. Meineke, vol. iv. p. 205. Bretschneider (*Geom. v. Euklid.*, p. 162) doubts the authenticity of this anecdote, and thinks that it may be only an imitation of the similar one concerning Euclid and Ptolemy. He does so on the ground that it is nowhere reported that Alexander had, besides Aristotle, Menaechmus as a special teacher in geometry. This is an insufficient reason for rejecting the anecdote, and, indeed, it seems to me that the probability lies in the other direction, for we shall see that Aristotle had direct relations with the school of Cyzicus.

[3] The fragments of Menaechmus have been collected and given in Greek by Max C. P. Schmidt (*l. c.*).

[4] Procl., *Comm.* ed. Friedlein, p. 67.

[5] *Ibid.*, p. 72.

[6] *Ibid.*, pp. 253–4.

the division of mathematical propositions into problems and theorems, he says, that whilst in the view of Speusippus and Amphinomus and their followers all propositions were theorems, it was maintained on the contrary by Menaechmus and the mathematicians of his School (οἱ περὶ Μέναιχμον μαθηματικοί) that they should all be called problems—the difference being only in the nature of the question stated, the object being at one time to find the thing sought, at another time, taking a definite thing, to see either what it is, or of what kind it is, or what affection it has, or what relation it has to something else.[7]

(*e*). In a fourth passage Proclus mentions him as the discoverer of the conic sections. The passage is in many respects so interesting that it deserves to be quoted in full.

'Again, Geminus divides a line into the compound and the uncompounded—calling a compound that which is broken and forms an angle; then he divides a compound line into that which makes a figure, and that which may be produced *ad infinitum*, saying that some form a figure, *e. g.* the circle, the ellipse (θυρεός),[8] the cissoid, whilst others do not form a figure, *e. g.* the section of the right-angled cone [the parabola], the section of the obtuse-angled cone [the hyperbola], the conchoid, the straight line, and all such. And again, after another manner, of the uncompounded line one kind is simple and the other mixed; and of the simple,

[7] *Ibid.*, pp. 77, 78.

[8] 'ὁ θυρεός (the door-shape, oblong; cf. Heron Alexandr., ed. Hultsch, *Definit.* 95, p. 27: ποιοῦσα σχῆμα θυροειδές). It is called by Eutocius, *Comm.* to Apollon. p. 10: ἔλλειψιν, ἣν καὶ θυρεὸν καλοῦσι, and is used several times in Proclus.' So Heiberg, who adds that in one passage it occurs in an extract from Eudemus, and says that we may perhaps assume that we have here the original name for the ellipse (*Nogle Puncter af de graeske Mathimatikeres Terminologi*, Philologisk-historiske Sam funds Mindeskrift, Kjobenhavn, 1879, p. 7). With relation to the same term, Heiberg, in his *Litterargeschichtliche Studien über Euklid*, Leipzig, 1882, p. 88, quotes a passage of the Φαινόμενα of Euclid which had hitherto been overlooked: ἐὰν γὰρ κῶνος ἢ κύλινδρος ἐπιπέδῳ τμηθῇ μὴ

one forms a figure, as the circular; but the other is indefinite, as the straight line; but of the mixed, one sort is in planes, the other in solids; and of that in planes, one kind meets itself as the cissoid, another may be produced to infinity; but of that in solids, one may be considered in the sections of solids, and the other may be considered as [traced] around solids. For the helix, which is described about a sphere or cone, exists around solids, but the conic sections and the spirical are generated from such a section of solids. But as to these sections, the conics were conceived by Menaechmus, with reference to which Eratosthenes says—

> 'Nor cut from a cone the Menaechmian triads';

but the latter [the spirics] were conceived by Perseus, who made an epigram on their invention:

> 'Perseus found the three [spirical] lines in five sections,
> and in honour of the discovery sacrificed to the gods.'

'But the three sections of the cone are the parabola, the hyperbola, and the ellipse; but of the spirical sections one kind is inwoven, like the *hippopede*;[9] and another kind is

παρὰ τὴν βασιν, ἡ τομὴ γίγνεται ὀξυγωνίου κώνου τομή, ἥτις ἐστὶν ὁμοία θυρεῷ, ed. D. Gregory, p. 561; and says that *θυρεός* was probably the name by which the curve was known to Menaechmus. It may be observed, however, that an ellipse is not of the shape of a door, neither is a shield, which is a secondary signification of *θυρεός*; the primary signification of the word is not 'door', but 'large stone' which might close the entrance to a cave, as in Homer (*Odyssey*, ix.); such a stone, or boulder, as may be met with on exposed beaches is often of a flattened oval form, and the names of a shield of such a shape, and of an ellipse, may have been thence derived.

[9] *τῶν δὲ σπειρικῶν τομῶν ἡ μέν ἐστιν ἐμπεπλεγμένη, ἐοικυῖα τῇ τοῦ ἵππου πέδῃ.* The *hippopede* is also referred to in the two following passages of Proclus *ἡ ἱπποπέδη, μία τῶν σπειρικῶν οὖσα* (ed. Fried. p. 127), and *καίτοιγε ἡ κισσοειδὴς μία οὖσα ποιεῖ γωνίαν καὶ ἡ ἱπποπέδη* (*ibid.* p. 128). In HERMATHENA, vol. v. p. 227, I said that a passage in Xenophon, *De re equestri*, cap. 7, explains why the name *hippopede* was given to the curve conceived by Eudoxus for the explanation of the motions of the planets, and in particular their

dilated in the middle, and becomes narrow at each extremity; and another being oblong, has less distance in the middle, but is dilated on each side.'[10]

(*f*). The line from Eratosthenes, which occurs in the preceding passage, is taken from the epigram which closes his famous letter to Ptolemy III., and which has been already more than once referred to. We now cite it with its context.

> μηδὲ σύ γ' Ἀρχύτεω δυσμήχανα ἔργα κυλίνδρων
> μηδὲ Μενεχμείους κωνοτομεῖν τριάδας
> δίζηαι, . . .[11]

(*g*). In the letter itself the following passage, which has

retrograde and stationary appearances, and also to one of the *spirics* of Perseus, each of which curves has the form of the lemniscate. The passage in Xenophon is as follows:—'Ἱππασίαν δ' ἐπαινοῦμεν τὴν πέδην καλουμένην· ἐπ' ἀμφοτέρας γὰρ τὰς γνάθους στρέφεσθαι ἐθίζει. Καὶ τὸ μεταβάλλεσθαι δὲ τὴν ἱππασίαν ἀγαθὸν, ἵνα ἀμφότεραι αἱ γνάθοι κατ' ἑκάτερον τῆς ἱππασίας ἰσάζωνται. Ἐπαινοῦμεν δέ καὶ τὴν ἑτερομήκη πέδην μᾶλλον τῆς κυκλοτεροῦς. *Ibid.* cap. 3. Τούς γε μὴν ἑτερογνάθους μηνύει μὲν καὶ ἡ πέδη καλουμένη ἱππασία, . . . This curve was named πέδη from its resemblance to the form of the loop of the wire in a snare, which was in fact that of a figure of 8. Some writers have given a different, and, to me it seems, not a correct, interpretation of the origin of this term. Mr. Gow, for example (*A Short History of Greek Mathematics*, Cambridge, 1884, p. 184), says: 'Lastly, Eudoxus is reported to have invented a curve which he called ἱπποπέδη, or "horse fetter," and which resembled those hobbles which Xenophon describes as used in the riding school.' In the next page Mr. Gow says: 'Eudoxus somehow used this curve in his description of planetary motions, . . .' This is not correct: the two curves were of a similar form—that of the lemniscate—and, therefore, the same name was given to each; but they differed widely geometrically, and were quite distinct from each other. See Knoche and Maerker, *Ex Procli successoris in Euclidis elementa commentariis definitionis quartae expositionem quae de recta est linea et sectionibus spiricis commentati sunt* J. H. Knochius et F. J. Maerkerus, Herefordiae, 1856, p. 14 *et seq.*; and Schiaparelli, *Le Sfere Omocentriche di Eudosso, di Callippo e di Aristotele*, Milano, 1875, p. 32 *et seq.*

[10] Procl. *Comm.* pp. 111, 112.

[11] Archimedes, ex. rec. Torelli, p. 146; Archim., *Opera*, ed. Heiberg, vol. iii., p. 112.

been already quoted (*Hermathena*, vol. v., p. 195), is found:

'The Delians sent a deputation to the geometers who were staying with Plato at Academia, and requested them to solve the problem [of the duplication of the cube] for them. While they were devoting themselves without stint of labour to the work, and trying to find two mean proportionals between the two given lines, Archytas of Tarentum is said to have discovered them by means of his semi-cylinders, and Eudoxus by means of the so-called *curved lines*. It was the lot of all these men to be able to solve the problem with satisfactory demonstration, while it was impossible to apply their methods practically so that they should come into use; except, to some small extent and with difficulty, that of Menaechmus.' [12]

(*h*). The solution of the *Delian Problem* by Menaechmus is also noticed by Proclus in his *Commentary on the Timaeus of Plato:*—'How then, two straight lines being given, it is possible to determine two mean proportionals, as a conclusion to this discussion, I, having found the solution of Archytas, will transcribe it, choosing it rather than that of Menaechmus, because he makes use of the conic lines, and also rather than that of Eratosthenes, because he employs the application of a scale.' [13]

(*i*). The solutions of Menaechmus—of which there are two—have been handed down by Eutocius in his Commentary on the Second Book of the Treatise of Archimedes *On the Sphere and Cylinder*, and will be given at length below.[14]

[12] *Ibid.* ex. rec. Torelli, p. 144; *ibid.* ed. Heiberg, vol. iii. pp. 104, 106.

[13] Procl. *in Platonis Timaeum*, p. 149 in libro iii. (ed. Joann. Valder, Basel, 1534). I have taken this quotation and reference from Max C. P. Schmidt, *Die fragmente des Mathematikers Menaechmus*, Philologus, xlii. p. 75. Heiberg (Archim. *Opera*, vol. iii. Praefatio v.) also gives this passage, but his reference is to p. 353, ed. Schneider.

[14] Archim., ed. Torelli, pp. 141 *et seq.*; Archim., *Opera*, ed. Heiberg, vol. iii. pp. 92 *et seq.*

(*j*). We learn from Plutarch that 'Plato blamed Eudoxus, Archytas, and Menaechmus, and their School, for endeavouring to reduce the duplication of the cube to instrumental and mechanical contrivances; for in this way [he said] the whole good of geometry is destroyed and perverted, since it backslides into the things of sense, and does not soar and try to grasp eternal and incorporeal images; through the contemplation of which God is ever God'.[15]

The same thing is repeated by Plutarch in his *Life of Marcellus* as far as Eudoxus and Archytas are concerned, but in this passage Menaechmus, though not mentioned by name, is, it seems to me, referred to. The passage is:—'The first who gave an impulse to the study of mechanics, a branch of knowledge so prepossessing and celebrated, were Eudoxus and Archytas, who embellish geometry by means of an element of easy elegance, and underprop by actual experiments and the use of instruments, some problems, which are not well supplied with proof by means of abstract reasonings and diagrams. That problem (for example) of two mean proportional lines, which is also an indispensable element in many drawings:—and this they each brought within the range of mechanical contrivances, by applying certain instruments for finding mean proportionals (μεσογράφους) taken from curved lines and sections (καμπύλων γραμμῶν καὶ τμημάτων). But, when Plato inveighed against them with great indignation and persistence as destroying and perverting all the good there is in geometry, which thus absconds from incorporeal and intellectual to sensible things, and besides employs again such bodies as require much vulgar handicraft: in this way *mechanics* was dissimilated and expelled from geometry, and being for a long

[15] Plut. *Quaest. Conviv.* lib. viii. *q*. 2, 1; Plut. *Opera*, ed. Didot, vol. iv. p. 876.

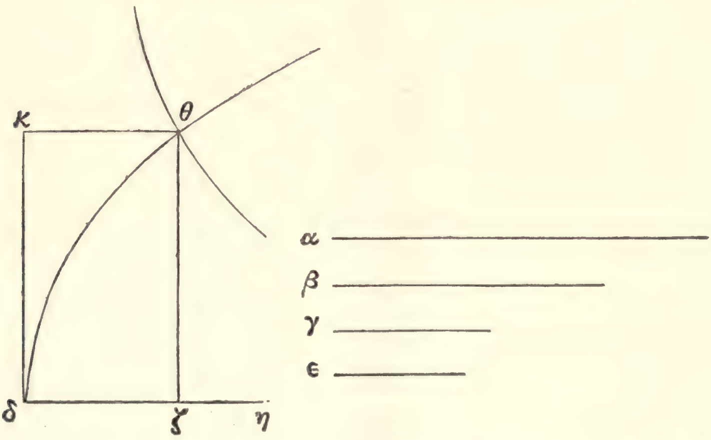
time looked down upon by philosophy, became one of the arts of war.'[16]

(*k*). Theon of Smyrna relates that 'he [Plato] blames those *philosophers* who, identifying the stars, as if they were inanimate, with spheres and their circles, introduce a multiplicity of spheres, as Aristotle thinks fit to do, and amongst the *mathematicians*, Menaechmus and Callippus, who introduced the system of deferent and restituent spheres (οἱ τὰς μὲν φερούσας, τὰς δὲ ἀνελιττούσας εἰσηγήσαντο).' [17]

The solutions of Menaechmus referred to in (*i*) are as follows :—

'As Menaechmus.

'Let the two given straight lines be α, ϵ; it is required to find two mean proportionals between them :—



'Let it be done, and let them be β, γ: and let the

[16] Ibid. *Vita Marcelli*, c. 14, sec. 5; Plut. *Opp.*, ed. Didot, vol. i. pp. 364, 5. The words κ. γ. in this passage refer to the curves of Eudoxus (see HERMATHENA, vol. v. pp. 217 and 225); τμ. refers to the solution of Archytas, and also, in my judgment, to the conic sections. Instead of τμ. we should, no doubt, expect to meet τομῶν; but Plutarch was not a mathematician, and the word, moreover, occurs in a biographical work: to this may be added, that in one of the *Definitions* of Heron (*Def.* 91, p. 26, ed. Hultsch), we find τμῆμα used for section.

[17] Theonis Smyrnaei Platonici *Liber de Astronomia*, ed. Th. H. Martin,

straight line $\delta\eta$, given in position and limited in δ, be laid down; and at δ let $\delta\zeta$, equal to the straight line γ, be placed on it, and let the line $\theta\zeta$ be drawn at right angles, and let $\zeta\theta$, equal to the line β, be laid down: since, then, the three straight lines a, β, γ are proportional, the rectangle under the lines a, γ, is equal to the square on β: therefore the rectangle under the given line a and the line γ, that is the line $\delta\zeta$, is equal to the square on the line β, that is to the square on the line $\zeta\theta$; therefore the point θ lies on a parabola described through δ. Let the parallel straight lines $\theta\kappa$, $\delta\kappa$ be drawn: since the rectangle under β, γ is given (for it is equal to the rectangle under a, ϵ), the rectangle $\kappa\theta\zeta$ is also given: the point θ, therefore, lies on a hyperbola described with the straight lines $\kappa\delta$, $\delta\zeta$ as asymptotes. The point θ is therefore given; so also is the point ζ.

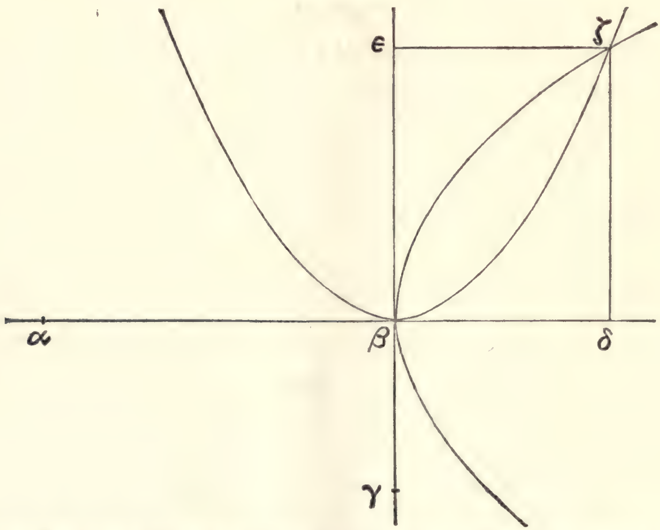
'The synthesis will be as follows:—

'Let the given straight lines be a, ϵ, and let the line $\delta\eta$ be given in position and terminated at δ; through δ let a parabola be described whose axis is $\delta\eta$ and parameter a. And let the squares of the ordinates drawn at right angles to $\delta\eta$ be equal to the rectangles applied to a, and having for breadths the lines cut off by them to the point δ. Let it [the parabola] be described, and let it be $\delta\theta$, and let the line $\delta\kappa$ [be drawn and let it] be a perpendicular; and with

pp. 330, 332, Paris, 1849. The σφαῖραι ἀνελίττουσαι were, according to this hypothesis, spheres of opposite movement, which have the object of neutralising the effect of other enveloping spheres (Aristot. Met. xii. c. 8, ed. Bekker, p. 1074). This modification of the system of concentric spheres of Eudoxus is attributed to Aristotle, but we infer from this passage of Theon of Smyrna that it was introduced by Menaechmus (Theon. Smyrn. *Liber de Astron.* Dissertatio, p. 59). Simplicius, however, in his *Commentary* on Aristotle *De Caelo* (*Schol.* in Aristot. Brandis, p. 498, *b*), ascribes this modification to Eudoxus himself. Martin (*l. c.*) thinks it probable that this hypothesis was put forward by Menaechmus, in his work on Plato's *Republic*, with reference to the description of the distaff of the Fates in the tenth book.

the straight lines $\kappa\delta$, $\delta\zeta$ as asymptotes, let the hyperbola be described, so that the lines drawn from it parallel to the lines $\kappa\delta$, $\delta\zeta$ shall form an area equal to the rectangle under α, ϵ: it [the hyperbola] will cut the parabola: let them cut in θ, and let perpendiculars $\theta\kappa$, $\theta\zeta$, be drawn. Since, then, the square on $\zeta\theta$ is equal to the rectangle under α and $\delta\zeta$, there will be: as the line α is to $\zeta\theta$, so is the line $\zeta\theta$ to $\zeta\delta$. Again, since the rectangle under α, ϵ is equal to the rectangle $\theta\zeta\delta$, there will be: as the line α is to the line $\zeta\theta$, so is the line $\zeta\delta$ to the line ϵ: but the line α is to the line $\zeta\theta$, as the line $\zeta\theta$ is to $\zeta\delta$. And, therefore: as the line α is to the line $\zeta\theta$, so is the line $\zeta\theta$ to $\zeta\delta$, and the line $\zeta\delta$ to ϵ. Let the line β be taken equal to the line $\theta\zeta$, and the line γ equal to the line $\delta\zeta$; there will be, therefore: as the line α is to the line β, so is the line β to the line γ, and the line γ to ϵ: the lines α, β, γ, ϵ are, therefore, in continued proportion; which was required to be found.

OTHERWISE.



'Let $\alpha\beta$, $\beta\gamma$ be the two given straight lines [placed] at right angles to each other; and let their mean proportionals be $\delta\beta$, $\beta\epsilon$, so that, as the line $\gamma\beta$ is to $\beta\delta$, so is the

line $\beta\delta$ to $\beta\epsilon$, and the line $\beta\epsilon$ to $\beta\alpha$, and let the perpendiculars $\delta\zeta$, $\epsilon\zeta$ be drawn. Since then there is: as the line $\gamma\beta$ is to $\beta\delta$, so is the line $\beta\delta$ to $\beta\epsilon$, therefore the rectangle $\gamma\beta\epsilon$, that is, the rectangle under the given straight line [$\gamma\beta$] and the line $\beta\epsilon$ will be equal to the square on $\beta\delta$, that is [the square] on $\epsilon\zeta$: since then the rectangle under a given line and the line $\beta\epsilon$ is equal to the square on $\epsilon\zeta$, therefore the point ζ lies on a parabola described about the axis $\beta\epsilon$. Again, since there is: as the line $\alpha\beta$ is to $\beta\epsilon$ so is the line $\beta\epsilon$ to $\beta\delta$, therefore the rectangle $\alpha\beta\delta$, that is, the rectangle under the given straight line [$\alpha\beta$] and the line $\beta\delta$, is equal to the square on $\epsilon\beta$, that is [the square] on $\delta\zeta$; the point ζ, therefore, lies on a parabola described about the axis $\beta\delta$: but it [the point ζ] lies also on another given [parabola] described about [the axis] $\beta\epsilon$: the point ζ is therefore given; as are also the perpendiculars $\zeta\delta$, $\zeta\epsilon$: the points δ, ϵ are, therefore, given.

'The synthesis will be as follows:—

'Let $\alpha\beta$, $\beta\gamma$ be the two given lines placed at right angles to each other, and let them be produced indefinitely from the point β: and let there be described about the axis $\beta\epsilon$ a parabola, so that the square on any ordinate [$\zeta\epsilon$] shall be equal to the rectangle applied to the line $\beta\gamma$ with the line $\beta\epsilon$ as height. Again, let a parabola be described about $\delta\beta$ as axis, so that the squares on its ordinates shall be equal to rectangles applied to the line $\alpha\beta$. These parabolas cut each other: let them cut at the point ζ, and from ζ let the perpendiculars $\zeta\delta$, $\zeta\epsilon$ be drawn. Since then, in the parabola, the line $\zeta\epsilon$, that is, the line $\delta\beta$ has been drawn, there will be: the rectangle under $\gamma\beta$, $\beta\epsilon$ equals the square on $\beta\delta$: there is, therefore: as the line $\gamma\beta$ is to $\beta\delta$, so is the line $\delta\beta$ to $\beta\epsilon$. Again, since in the parabola the line $\zeta\delta$, that is, the line $\epsilon\beta$, has been drawn, there will be: the rectangle under $\delta\beta$, $\beta\alpha$ equals the

square on $\epsilon\beta$: there is, therefore: as the line $\delta\beta$ is to $\beta\epsilon$, so is the line $\beta\epsilon$ to $\beta\alpha$; but there was: as the line $\delta\beta$ is to $\beta\epsilon$, so is the line $\gamma\beta$ to $\beta\delta$: and thus there will be, therefore: as the line $\gamma\beta$ is $\beta\delta$, so is the line $\beta\delta$ to $\beta\epsilon$, and the line $\beta\epsilon$ to $\beta\alpha$; which was required to be found.'

Eutocius adds—'The parabola is described by means of a compass (διαβήτου) invented by Isidore of Miletus, the engineer, our master, and described by him in his Commentary on the Treatise of Heron *On Arches* (καμαρικῶν).'

We have, therefore, the highest authority—that of Eratosthenes, confirmed by Geminus, (*e*) and (*f*)—for the fact that Menaechmus was the discoverer of the three conic sections, and that he conceived them as sections of the cone. We see, further, that he employed two of them, the parabola and the rectangular hyperbola, in his solutions of the Delian Problem. We learn, however, from a passage of Geminus, quoted by Eutocius in his Commentary on the *Conics* of Apollonius, which has already been referred to in another connexion (HERMATHENA, vol. iii., p. 169), that these names, *parabola* and *hyperbola*, are of later origin, and were given to these curves by Apollonius:—

'But what Geminus says is true, that the ancients (οἱ παλαιοί), defining a cone as the revolution of a right-angled triangle, one of the sides about the right angle remaining fixed, naturally supposed also that all cones were right, and that there was one section only in each—in the right-angled one, the section now called a *parabola*, in the obtuse-angled, the *hyperbola*, and in the acute-angled the *ellipse*; and you will find the sections so named by them. As then the original investigators (ἀρχαίων) observed the two right angles in each individual kind of triangle, first in the equilateral, again in the isosceles, and lastly in the scalene; those that came after them proved the general theorem as follows:—"The three angles of every triangle

are equal to two right angles." So also in the sections of a cone; for they viewed the so-called "section of the right-angled cone" in the right-angled cone only, cut by a plane at right angles to one side of the cone; but the section of the obtuse-angled cone they used to show as existing in the obtuse-angled cone; and the section of the acute-angled cone in the acute-angled cone; in like manner in all the cones drawing the planes at right angles to one side of the cone; which also even the original names themselves of the lines indicate. But, afterwards, Apollonius of Perga observed something which is universally true—that in every cone, as well right as scalene, all these sections exist according to the different application of the plane to the cone. His contemporaries, admiring him on account of the wonderful excellence of the theorems of conics proved by him, called Apollonius the "*Great Geometer*." Geminus says this in the sixth book of his *Review of Mathematics*.'[18]

The statement in the preceding passage as to the original names of the conic sections is also made by Pappus, who says, further, that these names were given to them by Aristaeus, and were subsequently changed by Apollonius to those which have been in use ever since.[19] In the writings of Archimedes, moreover, the conic sections are always called by their old names, and thus this statement of Geminus is indirectly confirmed.[20]

[18] Apollonii *Conica*, ed. Halleius, p. 9.

[19] Pappi Alexand. *Collect.* vii. ed. Hultsch, pp. 672 *et seq.* Mr. Gow (*Op. cit.*), p. 186, note, says: 'That Menaechmus used the name "section of right-angled cone," etc., is attested by Pappus, vii. (ed. Hultsch), p. 672.' This is not correct; the name of Menaechmus does not occur in Pappus.

[20] Heiberg (*Nogle Puncter af de graeske Mathematikeres Terminologi*, Kjobenhavn, 1879, p. 2) points out that 'Only in three passages is the word ἔλλειψις found in the works of Archimedes, but everywhere it ought to be removed as a later interpolation, as Nizze has already asserted.' These passages are: 1°. περὶ κωνοειδέων, ed. Torelli, p. 270, ed. Heiberg, vol. i.

It is much to be regretted that the two solutions of Menaechmus have not been transmitted to us in their original form. That they have been altered, either by Eutocius or by some author whom he followed, appears not only from the employment in these solutions of the terms parabola and hyperbola, as has been already frequently pointed out,[21] but much more from the fact that the language used in them is, in its character, altogether that of Apollonius.[22]

Let us now examine whether any inference can be drawn from the previous notices as to the way in which Menaechmus was led to the discovery of his curves. This question has been considered by Bretschneider,[23] whose hypothesis as to the course of the inquiry is very simple and quite in accordance with what we know of the state of geometry at that time.

We have seen that the right cone only was considered, and was conceived to be cut by a plane perpendicular to a side; it is evident, moreover, that this plane is at right angles to the plane passing through that side and the axis of the cone. We have seen, further, that if the vertical angle of the cone is right, the section is the curve, of which the fundamental property—expressed now by the equation

pp. 324, 325; 2°. *ibid.* Tor. p. 272, Heib. *id.* p. 332, l. 22; 3°. *ibid.* Tor. p. 273, Heib. *id.* p. 334, l. 5. Heiberg, moreover, calls attention to a passage where Eutocius (*Comm.* to Archimedes, *περὶ σφαίρας καὶ κυλίνδρου* II. ed. Tor. p. 163, ed. Heib. vol. iii. p. 154, l. 9) attributes to Archimedes a fragment he has discovered, containing the solution of a problem which requires the application of conic sections, among other reasons because in it their original names are used.

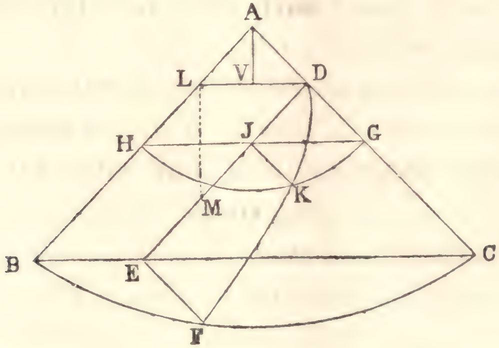
[21] First, as far as I know, by Reimer, *Historia problematis de cubi duplicatione*, Gottingae, 1798, p. 64, note.

[22] *e.g.* *παραβολή, ὑπερβολή, ἀσυμπτώτοις, ἄξων, ὀρθία πλευρά.* The original name for the asymptotes *αἱ ἔγγιστα* is met with in Archimedes, *De Conoidibus*, &c. (*αἱ ἔγγιστα τᾶς τοῦ ἀμβλυγωνίου κώνου τομᾶς*, ed. Heiberg, vol. i. p. 276, l. 22; and again, *αἱ ἔγγιστα εὐθεῖαι, κ.τ.λ.*, id. p. 278, l. 1). See Heiberg, *Nogle Punct.*, &c., p. 11.

[23] Bretsch. *Geom. v. Eukl.* pp. 156 *et seq.*

$y^2 = px$—was known to Menaechmus. This being premised, Bretschneider proceeds to show how this property of the parabola may be obtained in the manner indicated.

Let DEF be a plane drawn at right angles to the side AC of the right cone whose vertex is A, and circular base BFC; and let the triangle BAC (right-angled at A) be the section of the cone made by the plane drawn through AC and the axis of the cone. Let the plane DEF cut the cone in the curve DKF, and the plane BAC in the line DE. If, now, through any point J of the line DE a plane HKG be drawn parallel to the base BFC of the cone, the section of the cone made by this plane will be a circle, whose plane will be at right angles to the plane BAC; to which plane the plane of the section DKF is also perpendicular; the



line JK of intersection of these two planes will then be at right angles to the plane BAC, and, therefore, to each of the lines HG and DE in that plane. Let now the line DL be drawn parallel to HG, and the line LM at right angles to LD. In the semicircle HKG the square on JK is equal to the rectangle HJG, that is, to the rectangle under LD and JG, or, on account of the similar triangles JDG and DLM, to the rectangle under DJ and DM. The section of the right-angled cone, therefore, is such that the square on the ordinate KJ is equal to the rectangle under a given line DM and the abscissa DJ.

Bretschneider proceeds then to the consideration of the sections of the acute-angled and obtuse-angled cones and investigates the manner in which Menaechmus may have been led to the discovery of properties similar to those which he had known in the semicircle, and found in the case of the section of the right-angled cone.

Let a plane be drawn perpendicular to the side AC of an acute-angled cone, and let it cut the cone in the curve DKE, and let the plane through AC and the axis cut the cone in the triangle BAC. Through any point J of the line DE let a plane be drawn parallel to the base of the cone, cutting the cone in the circle HKG, whose plane will be at right angles to the plane BAC, to which plane the plane of the section DKE is also perpendicular. The line JK of intersection of these two planes will then be at right angles to the plane BAC; and, therefore, to each of the lines HG and DE in that plane, draw LD and EF parallel to HG, and at the point L draw a perpendicular to LD, intersecting DE in the point M. We have then

$$HJ : JE :: LD : DE$$

$$JG : JD :: EF : DE;$$

therefore,

$$HJ \, . \, JG : JE \, . \, JD :: LD \, . \, EF : DE^2.$$

But, on account of the similar triangles DEF and DLM,

$$EF : DE :: MD : LD.$$

Hence we get

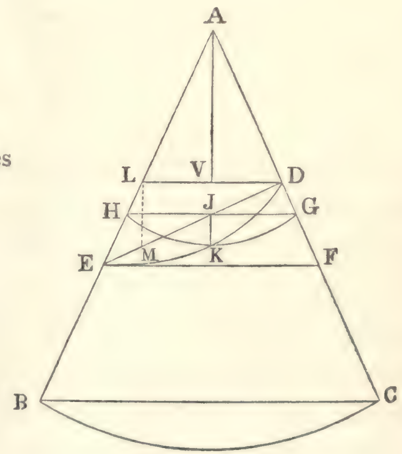
$$HJ \, . \, JG : JE \, . \, JD :: MD : DE.$$

But in the semicircle HKG

$$JK^2 = HJ \, . \, JG;$$

therefore,

$$JK^2 : JE \, . \, JD :: MD : DE,$$



that is, the square of the ordinate JK is to the rectangle under EJ and JD in a constant ratio.

The investigation in the case of the section of the obtuse-angle cone is similar to the above.

Bretschneider observes that the construction given for MD in the preceding investigations is so closely connected with the position of the plane of section DKE at right angles to the side AC that it could scarcely have escaped the observation of Menaechmus.

This hypothesis of Bretschneider, as to the properties of the conic sections first perceived by Menaechmus, which properties he employed to distinguish his curves from each other, seems to me to be quite in accordance as well with the state of geometry at that time as with the place which Menaechmus occupied in its development.

A comparison of these investigations with the solution of Archytas (see HERMATHENA, vol. v. p. 196, and *seq.*) will show, as there stated, that 'the same conceptions are made use of, and the same course of reasoning is pursued' in each (*id.* p. 199):

In each investigation two planes are perpendicular to an underlying plane; and the intersection of the two planes is a common ordinate to two curves lying one in each plane. In one of the intersecting planes the curve is in each case a semicircle, and the common ordinate is, therefore, a mean proportional between the segments of its diameter. So far the investigation is the same for all. Now, from the consideration of the figure in the underlying plane—which is different in each case—it follows that:—in the first case—the solution of Archytas—the ordinate in the second intersecting plane is a mean proportional between the segments of its base, whence it is inferred that the extremity of the ordinate in this plane also lies on a semicircle; in the second case—the section of the right-angled cone—the ordinate is a mean proportional between a given straight line and the abscissa; and, lastly, in the third case—the section of an acute-

angled cone—the ordinate is proportional to the geometric mean between the segments of the base.

So far, it seems to me, we can safely go, but not farther. From the first solution of Menaechmus, however, it has been generally inferred that he must have discovered the asymptotes of the hyperbola, and have known the property of the curve with relation to these lines, which property we now express by the equation $xy = a^2$. Menaechmus may have discovered the asymptotes; but, in my judgment, we are not justified in making this assertion, on account of the fact, which is undoubted, that the solutions of Menaechmus have not come down to us in his own words. To this may be added that the words *hyperbola* and *asymptotes* could not have been used by him, as these terms were unknown to Archimedes.

From the passage in the letter of Eratosthenes at the end of extract (*g*), coupled with the statement of Plutarch (*j*), Bretschneider infers that it is not improbable that Menaechmus invented some instrument for drawing his curves.[24] Cantor considers this interpretation as not impossible, and points out that there is in it no real contradiction to the observation in Eutocius concerning the description of the parabola by Isidore of Miletus.[25] Bretschneider adds that if Menaechmus had found out such an instrument it could never have been in general use, since not the slightest further mention of it has come down to us. It appears to me, however, that it is more probable that Menaechmus constructed the parabola and hyperbola by points, though this supposition is rejected by Bretschneider on the ground that such a construction would be very tedious. On the other hand, it seems to me that the words of Eratosthenes would apply very well to such a procedure. We know, on the authority of Eudemus (see HERMATHENA, vol. iii.,

[24] *Ibid.* p. 162.

[25] *Geschich. der Math.* p. 211.

p. 181), that 'the inventions concerning the application of areas'—on which, moreover, the construction by points of the curves $y^2 = p\,x$ and $x\,y = a^2$ depend—'are ancient ἀρχαῖα, and are due to the Pythagoreans':[26] it may be fairly inferred, then, that problems of application were frequently solved by the Greeks. And we have the very direct testimony of Proclus in the passage referred to, that the inventors of these constructions applied them also to the arithmetical solution of the corresponding problems. It is not surprising, therefore, to find—as Paul Tannery[27] has remarked—Diophantus constantly using the expression παραβάλλειν παρὰ in the sense of dividing.[28]

[26] Procl. *Comm.* ed. Fried. p. 419.

[27] *De la Solution Géométrique des Problèmes du Second Degré avant Euclide* (Mémoires de la Société des Sciences phys. and nat. de Bordeaux, t. iv., 2e Série, 3e Cahier, p. 409. Tannery (Bulletin des Sc. Math. et Astron. Tom. IV., 1880, p. 309) says that we must believe that Menaechmus made use of the properties of the conic sections, which are now expressed by the equation between the ordinate and the abscissa measured from the vertex, for the construction of these curves by points.

[28] In a Paper published in the Philologus (*Griechische und römische mathematik*, Phil. XLIII, 1884, pp. 474, 5), Heiberg puts forward views which differ widely from those stated above. He holds:—that it is not certain that Menaechmus contrived an apparatus for the delineation of the conic sections: that the only meaning which can be attached to Plato's blame (*j*) is, that Archytas, Eudoxus, and Menaechmus had employed, for the duplication of the cube, curves which could not be constructed with the rule and compass; and that the passage of Eratosthenes merely says that the curves of Menaechmus could be constructed, and not that he had found an apparatus for the purpose. Heiberg says, moreover, that it cannot be doubted that the Pythagoreans solved, by means of the application of areas, the equations, which we now call the vertical equations of the conic sections: but while admitting this, he holds that there is no ground for inferring thence that these equations were employed for the description of the conic sections by points; and says that such a description by points runs counter to the whole spirit of Greek geometry. On the other hand it seems to me that Tannery is right in believing that the *quadratrix* of Dinostratus (the brother of Menaechmus), or of Hippias, the contemporary of Socrates, was constructed in this manner (see Bulletin des Sc. Math. et Astron. *Pour l'histoire des lignes and Surfaces Courbes dans l'Antiquité*, t. VII. p. 279). Moreover, the construction of the para-

The extracts from Proclus (*b*), (*c*) and (*d*) are interesting as showing that Menaechmus was not only a discoverer in geometry, but that questions on the philosophy of mathematics also engaged his attention.

In the passages (*c*) and (*d*), moreover, the expression *οἱ περὶ Μέναιχμον μαθηματικοί* occurs—precisely the same expression as that used by Iamblichus with reference to Eudoxus (see HERMATHENA, vol. v. p. 219)—and we observe that in (*d*) this expression stands in contrast with *οἱ περὶ Σπεύσιππον*, which is met with in the same sentence. From this it follows that Menaechmus had a school, and that it was looked on as a *mathematical* rather than as a *philosophical* school. Further, we have seen that Theon of Smyrna makes a similar distinction between Aristotle on the one side and Menaechmus and Callippus on the other (*k*). Lastly, we learn from Simplicius that Callippus of Cyzicus, who was the pupil of Polemarchus, who was known to, or rather the friend of (*γνωρίμῳ*), Eudoxus, went with Polemarchus to Athens, in order to hold a conference with Aristotle on the inventions of Eudoxus, in order to rectify and perfect them.[29]

When these statements are put together, and taken

bola and rectangular hyperbola by points depends on the simplest problems of application of areas — the *παραβολή* without the addition of the *ὑπερβολή* or *ἔλλειπσις*.

[29] The passage is in the *Commentary* of Simplicius on the second book of Aristotle, *De Caelo*, and is as follows:— *εἴρηται καὶ ὅτι πρῶτος Εὔδοξος ὁ Κνίδιος ἐπέβαλε ταῖς διὰ τῶν ἀνελιττουσῶν καλουμένων σφαιρῶν ὑποθέσεσι, Κάλλιππος δὲ ὁ Κυζικηνὸς Πολεμάρχῳ συσχολάσας τῷ Εὐδόξου γνωρίμῳ, καὶ μετ' ἐκεῖνον εἰς Ἀθήνας ἐλθών, τῷ Ἀριστοτέλει συγκατεβίω, τὰ ὑπὸ τοῦ Εὐδόξου εὑρεθέντα σὺν τῷ Ἀριστοτέλει διορθούμενος τε καὶ προσαναπληρῶν.*—*Scholia* in Aristot. Brandis, p. 498, *b*. Callippus and Polemarchus, as Böckh has remarked, could not have been fellow-pupils of Eudoxus: Callippus, who flourished *circ.* 330 B.C., was too young. The meaning of the passage must be as stated above. Böckh conjectures that Polemarchus was about twenty years older than Callippus. See *Sonnenkreise*, p. 155.

in conjunction with the fact mentioned by Ptolemy, that Callippus made astronomical and meteorological observations at the Hellespont,[30] we are, I think, justified in assuming that the reference in each is to the School of Cyzicus, founded by Eudoxus, whose successors were—Helicon (probably), Menaechmus, Polemarchus, and Callippus.

From the passages of Plutarch referred to in (*j*) we see that Plato blamed Archytas, Eudoxus and Menaechmus for reducing the duplication of the cube to mechanical contrivances. On the other hand the solution of this problem, attributed to Plato, and handed down by Eutocius, is purely mechanical. Grave doubts have arisen hence as to whether this solution is really due to Plato. These doubts are increased if reference be made to the following authorities:—

Eratosthenes, in his letter in which the history of the Delian problem is given, refers to the solutions of Archytas, Eudoxus, and Menaechmus, but takes no notice of any solution by Plato, though mentioning him by name; Theon of Smyrna also, quoting a writing of Eratosthenes entitled 'The Platonic,' relates that the Delians sent to Plato to consult him on this problem, and that he replied that the god gave this oracle to the Delians, not that he wanted his altar doubled, but that he meant to blame the Hellenes for their neglect of mathematics and their contempt of geometry.[31] Plutarch, too, gives a similar account of the matter, and adds that Plato referred the Delians, who implored his aid, to Eudoxus of Cnidus, and Helicon of Cyzicus, for its solution.[32] Lastly, John Philoponus, in his

[30] *φάσεις ἀπλανῶν ἀστέρων καὶ συναγωγὴ ἐπισημασιῶν*, Ptolemy, ed. Halma, Paris, 1819, p. 53.

[31] Theon. Smyrn. *Arithm.* ed. de Gelder, Lugdun. Bat. 1827, page 5.

[32] Plutarch, *de Genio Socratis*, *Opera*, ed. Didot, vol. iii. p. 699.

account of the matter, agrees in the main with Plutarch, but in Plato's answer to the Delians he omits all reference to others.[33]

Cantor, who has collected these authorities, sums up the evidence, and says the choice lies between—1° the assumption that Plato, when blaming Archytas, Eudoxus, and Menaechmus, added, that it was not difficult to execute the doubling of the cube mechanically; that it could be effected by a simple machine, but that this was not geometry; or 2° the rejection, as far as Plato is concerned, of the communication of Eutocius, on the ground of the statements of Plutarch and the silence of Eratosthenes; or lastly, 3° the admission that a contradiction exists here which we have not sufficient means to clear up.[34]

The fact that Eratosthenes takes no notice of the solution of Plato seems to me in itself to be a strong presumption against its genuineness. When, however, this silence is taken in connexion with the statements of Plutarch, that Plato referred the Delians to others for the solution of their difficulty, and also that Plato blamed the solutions of the three great geometers, who were his contemporaries, as mechanical—a condemnation quite in accordance, moreover, with the whole spirit of the Platonic philosophy—we are forced, I think, to the conclusion that the sources from which Eutocius took his account of this solution are not trustworthy. This inference is strengthened by the fact, that the source from which the solution given by Eudoxus of the same problem was known to Eutocius, was so corrupt that it was unintelligible to him, and, therefore, not handed down by him.[35]

[33] Johan. Philop. *ad Aristot. Analyt. post.* i. 7.

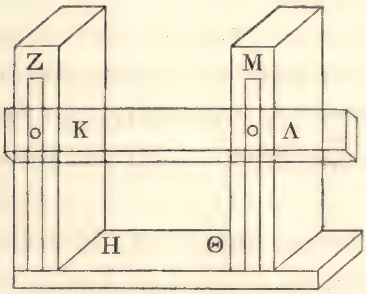
[34] Cantor, *Geschich. der Math.*, p. 202.

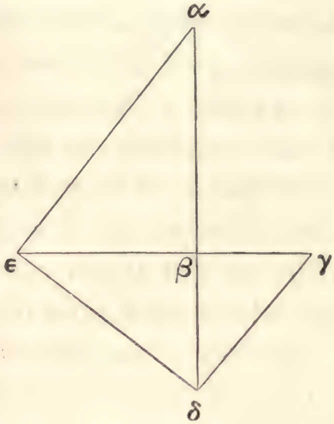
[35] See HERMATHENA, vol. v. p. 225.

The solution attributed to Plato is as follows :—

'As Plato.

'Two straight lines being given to find two mean proportionals in continued proportion.





'Let the two given straight lines $\alpha\beta$, $\beta\gamma$, between which it is required to find two mean proportionals, be at right angles to each other. Let them be produced to δ, ϵ. Now let there be constructed a right angle ZHΘ, and in either leg, as ZH, let a ruler KΛ be moved in a groove which is in ZH, so as to remain parallel to HΘ. This will take place if we imagine another ruler connected with ΘH and parallel to ZH, as ΘM. For the upper surfaces of the rulers ZH, ΘM being furrowed with grooves shaped like a dove-tail, in these grooves tenons connected with the ruler KΛ being inserted, the motion of the ruler KΛ will be always parallel to HΘ. This being arranged, let either leg of the angle, as HΘ, be placed in contact with the point γ, and let the angle and the ruler be moved so far that the point H may fall on the line $\beta\delta$, whilst the leg HΘ is in contact with the point γ, and the ruler KΛ be in contact with the line $\beta\epsilon$ at the point K, but on the other side with the point α: so that, as in the diagram, a right

angle be placed as the angle $\gamma\delta\epsilon$, but the ruler ΚΛ have the position of the line $\epsilon\alpha$. This being so, what was required will be done; for the angles at δ and ϵ being right, there will be the line $\gamma\beta$ to $\beta\delta$, as the line $\delta\beta$ to $\beta\epsilon$, and the line $\epsilon\beta$ to $\beta\alpha$.'[36]

The instrument is in fact a gnomon, or carpenter's square, with a ruler movable on one leg and at right angles to it, after the manner of a shoemaker's size-stick.

If this solution be compared with the second solution of Menaechmus it will be seen that the arrangement of the two given lines and their mean proportianals is precisely the same in each, and that, moreover, the analysis must also be the same. Further, a reference to the solution of Archytas (see HERMATHENA, vol. V. pp. 196 and 198 (*b*)) will show that the only geometrical theorems made use of in the solution attributed to Plato were known to Archytas. Hence it seems to me that it may be fairly inferred that this solution was subsequent to that of Menaechmus, as his solution was to that of Archytas. This, so far as it goes, is in favour of the first supposition of Cantor given above.

On account of the importance of the subject treated of here, I will state briefly my views on the matter in question:—Menaechmus was led by the study of the solution of Archytas, in the manner given above, to the discovery of the curve whose property (σύμπτωμα) is that now defined by the equation $y^2 = px$. Starting from this, he arrived at the properties of the sections of the acute-angled and of the obtuse-angled right cones, which are analogous to the well-known property of the semicircle—the ordinate is a mean proportional between the segments of the diameter. Having found the curve defined by the property, that its ordinate is a mean proportional between a given line and

[36] Archim. ed. Torelli, p. 135; Archim. *Opera*, ed. Heiberg, vol. iii. pp. 66 *et seq.* I have taken the diagrams used in this solution and that of Menaechmus from Heiberg's edition of Archimedes.

the abscissa, Menaechmus saw that by means of two such curves the problem of finding two mean proportionals could be solved, as given in the second of his two solutions, which, I think, was the one first arrived at by him. The question was then raised—Of what practical use is your solution? or, in other words, how can your curve be described?

Now we have seen in the former parts of this Paper that, side by side with the development of abstract geometry by the Greeks, the practical art of geometrical drawing, which they derived originally from the Egyptians, continued to be in use: that the Pythagoreans especially were adepts in it, and that, in particular, they were occupied with problems concerning the application (παραβολή) of areas, including the working of numerical examples of the same. Now any number of points, as near to each other as we please, on the curve $y^2 = px$, can be obtained with the greatest facility by this method; and in this manner, I think, Menaechmus traced the curve known subsequently by the name parabola—a name transferred from the *operation* (which was the proper signification of παραβολή) to the result of the operation. We have seen that the same name, παραβολή, was transferred and applied to division, which was also a transference of a name of an operation to its result.

Having solved the problem by the intersection of two parabolas, I think it probable that Menaechmus showed that the practical solution of the question could be simplified by using, instead of one of them, the curve $xy = a^2$, the construction of which by points is even easier than that of the parabola. There is no evidence, however, for the inference that Menaechmus knew that this curve was the same as the one he had obtained as a section of the obtuse-angled cone; or that he knew of the existence of the asymp totes of the hyperbola, and its equation in relation to them.

Let us examine now whether anything can be derived from the sources, which would enable us to fix the time of the Delian deputation to Plato—be it real or fictitious.

We have seen that Sotion, after mentioning that Eudoxus took up his abode at Cyzicus, and taught there and in the neigbouring cities of the Propontis, relates that subsequently he returned to Athens accompanied by a great many pupils (πάνυ πολλοὺς περὶ ἑαυτὸν ἔχοντα μαθητάς), for the sake, as some say, of annoying Plato, because formerly he had not held him worthy of attention (HERMATHENA, vol. v. pp. 213, 214). We learn, further, from Apollodorus that Eudoxus flourished about the hundred and third Olympiad—B.C. 367—and it is probable, as Böckh thinks, that this time falls in with his residence at Cyzicus. Now the narrative of Plutarch—that Plato referred the Delians to Eudoxus and Helicon for the solution of their difficulty—points to the time of the visit of Eudoxus and his pupils to Athens, for—1° as we know from Sotion, Plato, and Eudoxus had not been on good terms; and 2° it is not probable that, before this visit, Helicon, who was a native of Cyzicus and a pupil of Eudoxus, as we learn from the spurious 13th *Epistle of Plato*, had become famous or was known to Plato. Böckh assumes, no doubt rightly, that the visit of Eudoxus and his pupils to Athens, and their sojourn there, took place a few years later than Ol. 103, 1—B.C. 367; so that it occurred between the second and third visits of Plato to Sicily (368 B.C. and 361 B.C.).[37] To this time, therefore, he refers the remarkable living and working together at the Academy of eminent men, who were distinguished in mathematics and astronomy, according to the report of Eudemus as handed down by Proclus. Now, amongst those named there we find Eudoxus himself, his pupil

[37] Böckh, *Sonnenkreise*, &c., pp. 156, 157.

Menaechmus, Dinostratus—the brother of Menaechmus—and Athenaeus of Cyzicus;[38] to these must be added Helicon of Cyzicus—more distinguished as an astronomer than a mathematician—who was recommended to Dionysius by Plato,[39] and who was at the court of Dionysius in company with Plato at the time of his third visit to Syracuse.[40]

I quite agree with Böckh in thinking that all the pupils of Eudoxus and the citizens of Cyzicus, whom we find at Athens at that time—even though they are not expressly named as pupils of Eudoxus—belonged to the school of Cyzicus: and I have no doubt that to these illustrious Cyzicenians the fame of the Academy—so far at least as mathematics and astronomy are concerned—is chiefly due.[41] It is noteworthy that Aristotle, at the time of this visit, so famous and so important in consequence of the impetus thereby given to the mathematical sciences, had recently joined the Academy, and was then a young man; and it is easy to conceive the profound impression made by Eudoxus and his pupils on a nature like that of Aristotle; and an explanation is thus afforded as well of the great respect which he entertained for Eudoxus, as of the cordial relations which existed later

[38] See HERMATHENA, vol. iii., p. 163.

[39] *Epist.* Plat. xiii.

[40] Plutarch, *Dion.*

[41] Zeller says: 'Among the disciples of Plato who are known to us, we find many more foreigners than Athenians: the greater number belong to that eastern portion of the Greek world which since the Persian War had fallen chiefly under the influence of Athens. In the western regions, so far as these were at all ripe for philosophy, Pythagoreanism, then in its first and most flourishing period, most probably hindered the spread of Platonism, despite the close relation between the two systems' (*Plato and the Older Academy*, E. T. pp. 553 *seq.*). Zeller gives in a note a list of Plato's pupils, in which all the distinguished men of the School of Cyzicus are placed to the credit of the Academy.

between him and the mathematicians and astronomers of the school of Cyzicus.[42]

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[42] Aristotle was born in the year 384 B.C., and went to Athens 367 B.C.: after the death of Plato (B.C. 347) Aristotle left Athens and went to Atarneus in Mysia, where his friend Hermias was *dynast.* When he was there he may have renewed his relations with the distinguished men of the School of Cyzicus, which was not far distant. It is quite possible that Menaechmus may have been recommended as mathematical teacher to Alexander the Great by Aristotle; and we have seen that Polemarchus, who was known to Eudoxus, and Callippus of Cyzicus, who was a pupil of Polemarchus, went together to Athens to hold a conference with Aristotle on the hypothesis of Eudoxus, with the view of rectifying and completing it.

END OF VOL. V.